we note that it will be greater the smaller the value of H (i.e., the smaller the value of b). It can be shown by analysing (2.5) and (2.6) (with $\tau_0^* \equiv 0, \sigma_0^* \equiv 1$) that

$$g_2 \to 0, \int g_1 J_0 (\rho t) dt \to -h m_0 \beta_2 Q^{-1}, \rho < 1, b \to 0$$

Substituting into (2.4), we obtain as $b \to 0$

$$u_z (r; H) = p_0 h m_0 \beta (\lambda + 2\mu)(\lambda + \mu)^{-1}, r < R$$

Thus, for any H (i.e., for any b)

$$| u_z (r; H)| < 2 (1 - \nu)| p_0 | h m_0 \beta_2, \nu = {}^1/_2 \lambda (\lambda + \mu)^{-1} \quad (5.2)$$

For numerical data /6/ $\Delta p = -40$ MPa, $h = 600$ m, $R = 10^4$ m, $\beta_2 = 2 \times 10^{-3}$ (MPa)$^{-1}$, $\beta = 1.5 \times 10^{-5}$ (MPa)$^{-1}$, $m_0 = 0.05, \nu = 0.34$ we obtain $p_0 = -34.3$ MPa, from which $| u_z (r; H)| < 2.74$ m according to (5.2), which agrees with the approximate estimate obtained in /7/.

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A GEOMETRICAL METHOD OF SOLVING THE PROBLEM OF MAXIMIZING THE NORM OF THE STATE VECTOR OF THE SYSTEM IN A FINITE CONTROL INTERVAL*

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The problem of constructing controls which maximize the norm of the state vector of the system at the right-hand end of a fixed control interval is considered. A numerical method of determining the maxima is proposed, based on a geometrical approach. Local convergence of the algorithm is proved and the direction of the search for the global maximum is discussed. Results of numerical modelling are given.

The problem of maximizing the convex function J on a convex manifold of attainability discussed here, cannot be solved using traditional methods (for example, the method of minimum discrepancy and its modifications /1, 2/), since in the case of an equivalent minimization the functional J is not convex. This leads, in particular, to violation of the theorems of uniqueness of optimal control. Indeed (Fig.1), more than one point may exist belonging to the convex manifold of attainability $K(T)$ at the maximum distance from the origin of coordinates. At the same time, there exists a unique point belonging to $K(T)$ whose distance from the origin of coordinates is a minimum.

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1. Formulation of the problem. It is required to find the control $u(t)$ which, in a fixed interval $(0, T)$, transfers to the linear system

$$x^{\cdot} = A(t)x + B(t)u, \quad x \in E^n, \quad u \in \Omega_u \subset E^m \tag{1.1}$$

(Ω_u is the boundary set) from the known initial state $x(0) = x_0$ to the point $x(T)$ at the right-hand end of the trajectory, at the maximum distance from the origin of coordinates O.

At least one such point exists. Indeed, the set $K(T)$ is bounded (and also compact), i.e., it is completely contained within some sphere. Reducing the radius of this sphere we finally arrive at a sphere of radius R^* which has at least one common point with the boundary ∂K of the domain of attainability (Fig.1). This clearly shows that the optimal trajectories, i.e. the trajectories maximizing the functional $J = \| x(T) \|$, terminate at the points on ∂K, and this means that the optimal control should be sought amongst the extremal ones. All this implies that the maximizing control is not unique. We shall assume in this paper that the problem is solved if at least one such control has been found.

2. Local maximum. Before anything else, we note that the point x' of contact K between the supporting hyperplane Γ and normal vector g is determined by the solution of the problem $x' = \operatorname{argmax}(x, g)$ (here and henceforth the maximum is taken over all $x \in K(T)$).

Indeed, let us consider the scalar product

$$(x, g) = g^T \Phi(T) x_0 + \int_0^T g^T \Phi(T, t) B(t) u(t)\, dt \tag{2.1}$$

where $\Phi(T, t)$ is the basic matrix solution of the homogeneous system. If Ω_u is a hypercube, i.e. $| u_i(t) | \leqslant 1$ $(i = 1, 2, \ldots, m)$, then the extremal control maximizing (2.1) will have the form

$$u(t) = \operatorname{sgn}[g^T \Phi(T, t) B(t)] \tag{2.2}$$

It is clear that the control generates the point x', since it is the vector drawn to the point of contact between the hyperplane and the boundary of the domain of attainability that has a maximum projection on the axis of the unit vector g (Fig.1).

The proposed algorithm is as follows: 1) the vector g^0 is chosen arbitrarily (the superscript indicates the number of the iteration), 2) the vector $x^i = \operatorname{argmax}(x, g^{i-1})$ is constructed, 3) the vector $g^i = x^i / \| x^i \|$ is constructed and step 2 of the algorithm is repeated, 4) the algorithm will terminate when the inequality $| \| x^i \| - \| x^{i-1} \| | \leqslant \varepsilon$, where ε is the given error, is satisfied. Step 2 is carried out by calculating $u(t)$ with help of formula (2.2), followed by integrating system (1.1) with this control.

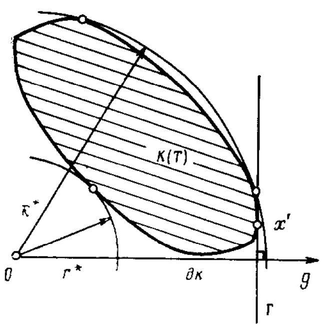

Fig.1

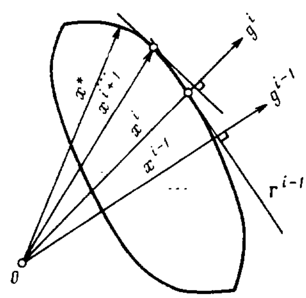

Fig.2

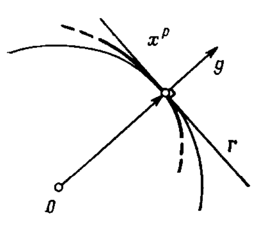

Fig.3

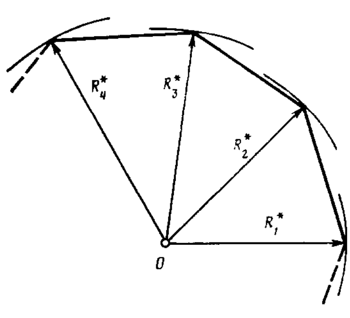

Fig.4

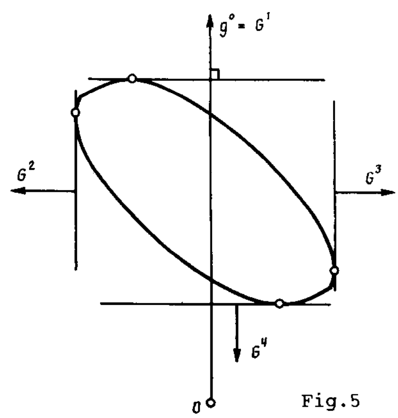

Fig.5

The geometrical content of the method is obvious. We construct for the vector x^{i-1} (Fig.2) a normal hyperplane Γ^{i-1}, and its point of contact with ∂K yields the vector x^i for the next iteration. A rigorous proof of the algorithm is given below.

Lemma 1. The process described above generates a sequence of vectors $\{x^i\}$ which converges with respect to the norm.

We shall first show that

$$\|x^i\| \geqslant \|x^{i-1}\|. \tag{2.3}$$

Indeed,

$$x^i = \operatorname{argmax}(x, g^{i-1}) = \operatorname{argmax}(x, x^{i-1}/\|x^{i-1}\|) = \operatorname{argmax}(x, x^{i-1}) \tag{2.4}$$

Let us consider the scalar product (x^i, x^{i-1}). If we take $x^i = x^{i-1}$, then the product will yield $\|x^{i-1}\|^2 = C_0$. Let us now assume that $\|x^i\| \leqslant \|x^{i-1}\|$. Then $(x^i, x^{i-1}) < C_0$ by the definition of the scalar product, and this means that x^i cannot satisfy condition (2.4). This is a contradiction, thus proving inequality (2.3). Having noted now that the quantity $\|x^i\|$ has an upper limit by virtue of the boundedness of $K(T)$, we arrive at the proof of the lemma.

We shall call the sequence constructed in this manner the *A*-sequence.

It is clear (Fig.2) that the algorithm converges to the point x^* at which the tangent hyperplane is normal to the vector x^*. Additional iterations from the point x^* give the same point x^*. It would be natural to assume that x^* is indeed a local maximum. In fact, this is not always true. An example could be constructed (Fig.3) in which the point x^p to which the algorithm converges "from the right" contains, in any of its ε-neighbourhoods, points for which $\|x\| > \|x^p\|$ represents a part of ∂K) "to the left" of the point x^p. In this case x^p is an unstable limit point of the algorithm (we shall call it the slippage point), since any point "to the left" of x^p and as close to x^p as desired, taken as the consecutive iteration step, will take the algorithm away from x^p. It is clear that in order for the point x^p to be indeed a local maximum, it is necessary that the algorithm converge to x^p from any point to the left.

Theorem 1. The point x^* will be a point of local maximum if and only if there exists a ε-neighbourhood $S_\varepsilon(x^*)$ of the point x^* such that the algorithm will converge to x^* from any point of the set $Q_\varepsilon = \partial K \cap S_\varepsilon(x^*)$ taken as the initial point.

We will first assume that x^* is a local maximum, i.e. that there exists a ε-neighbourhood of x^* such, that

$$\|x\| < \|x^*\|,\ \forall x \in P_\varepsilon = K(T) \cap S_\varepsilon(x^*) \tag{2.5}$$

We will choose an arbitrary point $x \in Q_\varepsilon$ and construct from it an *A*-sequence converging to some point x^{p1}. If $x^{p1} \neq x^*$, then x^{p1} will be a slippage point and $\|x^{p1}\| < \|x^*\|$. Let us take $\varepsilon_1 < \varepsilon$ so that $x^{p1} \notin Q_{\varepsilon 1}$ and again construct the *A*-sequence, and then continue this process. Then, either we shall obtain $x^{pi} = x^*$ at the *i*-th step and the theorem will be proved directly from $\varepsilon = \varepsilon_i$, or the following infinite sequence converging to x^* will be constructed:

$$\|x\| \leqslant \|x^{p1}\| \leqslant \|x^{p2}\| \leqslant \ldots \leqslant \|x^*\|.$$

Here the equalities correspond to the situation in which Q_ε represents a part of the sphere $S_{\|x^*\|}(0)$, and this is impossible by virtue of inequality (2.5). The infinite sequence of strict inequalities is also impossible here, since its existence would mean, by virtue of the fact that ε is infinitely small, that the distance from the point 0 to the slippage points could be as small as desired, i.e. the surface Q_ε will lie as near to that sphere as desired.

In order to prove the assertion in the other direction we take any point $x \in Q_\varepsilon$ and construct for it the *A*-sequence $\{x^i\}$. By virtue of the inequality (2.3) and the fact that $\|x^i\| \to \|x^*\|$, we obtain at once $\|x\| \leqslant \|x^*\|$, i.e. x^* is a local maximum.

We note that the theorem just proved does not provide an answer to whether we have an extremum at the point x^p, or whether it represents a slippage point, since it is impossible to inspect all points of the ε-neighbourhood for the "withdrawal" from x^p. However, the next section will show that there is no need to distinguish between the local maximum and a slippage point.

3. ***Global maximum.*** We have already mentioned that the maximizing control is not unique (Fig.1). Moreover, we can construct a convex curve (the angular curve in Fig.4) containing as many local extrema as desired, to every one of which the algorithm will converge by a suitable choice of the initial vector g^0. It is clear that the attempt to construct an algorithm for finding all maxima (and the subsequent choice of the global maximum) will be untenable. A more acceptable scheme would be that of organizing a passage from one local maximum to the next with a larger value of the functional. However, here we again cannot guarantee any progress towards a global extremum since the mutual positioning of the point *O* and the set $K(T)$ (depending, in particular, on the initial state of the system x_0), is arbitrary. Therefore, the author used the following heuristic algorithm: first, we choose an arbitrary unit

vector g^0 and use the Gram-Schmidt method to construct the system of $n-1$ orthonormed vectors. Further, we add to these n orthogonal vectors the same number of vectors of opposite direction and then use all $2n$ vectors as initial vectors (the plane case is illustrated in Fig.5). In numerical experiments carried out for $n=2$ we found no cases in which a set of local maxima obtained in this manner did not contain a global maximum. The algorithm converges to a local maximum after 2-4 iterations irrespective of the dimensions of the system (up to $n=6$ in the experiments), and this is at least twice as fast as in the case of similar methods /2/ of searching for the minimum of a functional.

In conclusion, we note that the proposed algorithm can be applied to non-linear systems including the case with a non-convex domain of attainability.

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ON THE CONSTRUCTION OF GENERAL SOLUTIONS OF THE THEORY OF THE ELASTICITY OF INHOMOGENEOUS SOLIDS*

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The elasticity theory equations are decomposed in the case when the shear modulus is a function of one Cartesian coordinate while Poisson's ratio is a function of three coordinates. Such a separation is possible for transverse isotropy when both shear coefficients depend just on the coordinates of the normal isotropy plane. It is assumed that the mass forces are potential.

Decomposition of the elasticity theory equations of an isotropic body by extraction of the normal rotation deformation /1/ was later extended to the case of a transversely-isotropic body /2/. Such a separation was performed for an isotropic body /3/ and for a transversely-isotropic body /4/* (*See also Puro, A.E., Some Exact Particular Solutions of the Statics Equations of an Inhomogeneous Medium, Candidate Dissertation, Tallinn, 1975.) for a one-dimensional inhomogeneity when the elasticity coefficients depend on one Cartesian coordinate.

1. A transversely-isotropic body is referred to a rectangular Cartesian system of coordinates and the z-axis is perpendicular to the plane of body isotropy.

We consider both shear coefficients $c_{44}=c^{-1}$, $(c_{11}-c_{12})/2=G$ in the generalized Hooke's law

$$\sigma_{xx}=c_{11}\varepsilon_{xx}+c_{12}\varepsilon_{yy}+c_{13}\varepsilon_{zz},\quad \sigma_{xy}=(c_{11}-c_{12})\varepsilon_{xy}$$
$$\sigma_{yy}=c_{12}\varepsilon_{xx}+c_{11}\varepsilon_{yy}+c_{13}\varepsilon_{zz},\quad \sigma_{xz}=2c_{44}\varepsilon_{xz}$$
$$\sigma_{zz}=\sigma=c_{13}(\varepsilon_{xx}+\varepsilon_{xy})+c_{33}\varepsilon_{zz},\quad \sigma_{yz}=2c_{44}\varepsilon_{yz}$$

differential functions of just the z coordinate while the remaining elasticity coefficients c_{ik} are functions of three coordinates. It is also assumed that the mass force vector **M** and the displacement vector **u** are decomposed into potential and solenoidal components in the plane of isotropy and expressed, respectively, in terms of the potentials

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